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TRIANGULAR REPRESENTATIONS OF SPLITTING RINGS

BY
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ABSTRACT. The term "splitting ring" refers to a nonsingular ring R such that for any right R-module M, the singular submodule of M is a direct summand of M. If R has zero socle, then R is shown to be isomorphic to a formal triangular matrix ring $\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, where A is a semiprime ring, C is a left and right artinian ring, and ${}_CB_A$ is a bimodule. Also, necessary and sufficient conditions are found for such a formal triangular matrix ring to be a splitting ring.

1. **Introduction and notation.** In [5, Theorem 10], we showed that any right nonsingular ring is an essential product of a ring with essential right socle and a ring with zero right socle. [An essential product of two rings is any subdirect product which contains an essential right ideal of the direct product.] Using this result, [5, Theorem 12] reduces the problem of characterizing splitting rings to characterizing those with either essential socle or zero socle. Since the case of essential socle has been taken care of by [4, Corollary 5.4], only the case of zero socle remains. The purpose of this paper is to study the structure of such a splitting ring with zero socle by representing the ring as a formal triangular matrix ring.

In this paper all rings are associative with identity, and all modules are unital. Unspecified modules are right modules; thus any statement about a bimodule ${}_CB_A$ refers to the module B_A unless the module ${}_CB$ is specifically mentioned. The reader is assumed to be familiar with the standard notions of singular and nonsingular modules; [3] or [4] may be consulted for details.

We give here for reference our notation, which coincides with that in [4]. For any ring R, we let $\mathcal{E}(R)$ denote the collection of essential right ideals of R. The singular submodule of a module A is denoted $Z(A)$, and for a ring R, we use $Z_r(R)$ in place of $Z(R_R)$. A submodule A of a module B is said to be *$\mathcal{E}$-closed* in B provided B/A is nonsingular, and we let $L^*(B)$ denote the collection of $\mathcal{E}$-closed submodules of B. Given any submodule A of B, there is a smallest $\mathcal{E}$-closed submodule C of B which contains A, and C is called the *$\mathcal{E}$-closure* of A in B. We note that the $\mathcal{E}$-closure of a two-sided ideal in a ring R is again a two-sided ideal of R. For ease of expression, a two-sided ideal of R which belongs to $L^*(R)$ is referred to as a "two-sided ideal in L^*(R)". Finally, we use S° (or S°_R) to stand for the localization functor associated with a right nonsingular ring R: S° is an exact functor from right R-modules to right modules over the ring $S^\circ R$ (which coincides with the maximal right quotient ring of R).

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2. The representation of a splitting ring. In this section, we represent a right nonsingular splitting ring R (with zero socle) as a formal triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$, where A is a semiprime ring and C is a left and right artinian ring. In order to accomplish this, some sort of chain condition on R is needed. This is provided by Theorem 1, which shows that the prime radical of R is finite dimensional. We conjecture that in fact R itself must be finite dimensional.

Theorem 1. *Let R be a right nonsingular splitting ring such that* $\operatorname{soc}(R_R) = 0$. *If P denotes the prime radical of R, then P_R is finite dimensional.*

Proof. Our general procedure is to show first that all nilpotent ideals of R are finite dimensional. This allows us to prove that P has plenty of finite-dimensional submodules, which we use to show that P itself is finite dimensional. Organizationally, the proof consists of a series of lemmas, some of which are proved in slightly more generality than needed here in order to be used in subsequent theorems. We stipulate that Lemmas C through J include the hypothesis that R is a right nonsingular splitting ring with $\operatorname{soc}(R_R) = 0$.

Lemma A. *Let $Z_r(R) = 0$, and let N be any nilpotent two-sided ideal of R. If M is the $\mathcal{S}$-closure of N_R in R_R, then M is a two-sided ideal in $L^*(R)$ whose left annihilator belongs to $\mathcal{S}(R)$.*

Proof. Letting H denote the left annihilator of M, we infer from the nonsingularity of R that H is also the left annihilator of N. To get $H \in \mathcal{S}(R)$, it suffices to show that any element $x \in R \setminus H$ has a nonzero right multiple in H. Observing that $xN \neq 0$, we infer that there must be a positive integer k for which $xN^k \neq 0$ and $xN^{k+1} = 0$. Choosing some $r \in N^k$ such that $xr \neq 0$, we see that xr is a nonzero element of H.

Lemma B. *Let A be any nonsingular right R-module with zero socle. If B is any submodule of A, then B is the intersection of those essential submodules of A which contain B.*

Proof. Given $x \in A \setminus B$, let K be maximal among those submodules of A which contain B but not x. We claim that K is essential in A.

Letting $f\colon A \to A/K$ denote the natural map, we see from the maximality of K that fx is a nonzero element of A/K which is contained in every nonzero submodule of A/K. Thus $(fx)R$ is a simple, essential submodule of A/K. Inasmuch as $\operatorname{soc}(A) = 0$, $(fx)R$ cannot be projective. Recalling that all simple modules are either singular or projective [4, pp. 55, 56], we see that $(fx)R$ must be singular. Since $(A/K)/(fx)R$ is singular also, we infer that A/K is singular. Inasmuch as A is nonsingular, we conclude that K is essential in A.

Lemma C. *Let M be a two-sided ideal in $L^*(R)$ whose left annihilator belongs to $\mathcal{S}(R)$. Then M_R is a direct summand of R_R.*

Proof. If H denotes the left annihilator of M, then H is a two-sided ideal in $\mathcal{S}(R)$. Inasmuch as R/M is a nonsingular right R-module, [4, Lemma 5.2] says that $(R/M)/(R/M)H$ is a projective right (R/H)-module. Then $R/(M + H)$ is projective as an (R/H)-module, whence $(M + H)/H$ is a direct summand of R/H. Thus there exists an element $m \in M$ such that $m^2 - m \in H$ and $mR + H = M + H$.

From the equation $m^2 - m \in H$, we obtain $m^3 - m \in H$. Since $m \in M$, this yields $m^4 - m^2 = 0$; hence the element $e = m^2$ is an idempotent. Observing that $eR + H = M + H$, we multiply this equation on the right by M to obtain $eM = M^2$, whence $M^2 = eR$. Thus M^2 is a direct summand of R_R; hence it suffices to show that $M = M^2$.

If $M \neq M^2$, then according to Lemma B, M_R must have a proper essential submodule K which contains M^2. We infer that $M/K = Z(R/K)$, from which it follows that M/K is a direct summand of R/K. Thus R must have a right ideal J such that $M + J = R$ and $M \cap J = K$. Observing that $M = M^2 + JM$, we obtain $M \leq J$, which leads to the contradiction $M = K$.

Lemma D. *Let e be an idempotent in the ring $Q = S^\circ R$ such that $R \cap eQ$ is a two-sided ideal of R.*

(a) *Re is a unital subring of eQe.*

(b) *Re is an essential right (Re)-submodule of eQe.*

(c) $Z_r(Re) = 0$ *and* $S^\circ(Re) = eQe$.

Proof. (a) We must show that Re is closed under multiplication, and that e is an identity for Re. Given any $x \in R$, we have $x(R \cap eQ) \leq R \cap eQ \leq eQ$. Since $R \cap eQ$ is essential in eQ, it follows that $x(eQ) \leq eQ$. Therefore $Re \subset eQ$, from which the required results are immediate.

(b) If not, then eQe contains a nonzero element t such that $Re \cap tRe = 0$.

Since tQ is nonzero, it cannot be a singular right R-module. Thinking of t as an endomorphism of Q_R, it follows that $\ker t$ cannot be essential in Q_R. Inasmuch as R_R is essential in Q_R, we infer that $R \cap t^{-1}R$ is essential in Q_R, and thus that $(R \cap t^{-1}R \cap eQ) \otimes (1 - e)Q$ is essential in Q_R. Therefore $(R \cap t^{-1}R \cap eQ) \otimes (1 - e)Q$ cannot be contained in $\ker t$. Noting that $(1 - e)Q \leq \ker t$ already, we see that $R \cap t^{-1}R \cap eQ \nleq R \cap \ker t$. In view of Lemma B, it follows that $R \cap t^{-1}R$ must have an essential submodule F which contains $R \cap \ker t$ but not $R \cap t^{-1}R \cap eQ$.

Set $I = \{(x, tx) | x \in F\}$ and $J = \{(x, tx) | x \in R \cap t^{-1}R\}$, both of which are submodules of R^2. Since F is essential in $R \cap t^{-1}R$, it follows that I is essential in J, whence J/I is singular. There exists a map f: $R^2 \to Q$ given by $f(x,y) = y - tx$, and we check that $\ker f = J$. Thus R^2/J is nonsingular; hence $J/I = Z(R^2/I)$. Therefore R^2 must contain a submodule K for which $J + K = R^2$ and $J \cap K = I$.

There exists an element $x \in (R \cap t^{-1}R \cap eQ)\backslash F$. Set $m_1 = x$ and $m_2 = tx$, and note that $m_1, m_2 \in R \cap eQ$. Also, let z_1 and z_2 denote the elements (1,0) and (0,1) in R^2.

Since $R^2 = J + K$, we obtain $z_i + (u_i, tu_i) \in K$ for some $u_i \in R \cap t^{-1}R$. Note that $z_i m_i + (u_i m_i, tu_i m_i) \in K$ also. Observing that $tu_i \in R \cap tR$, we obtain $tu_i e \in Re \cap tRe = 0$, and thus $tu_i m_i = tu_i e m_i = 0$. Therefore $u_i m_i \in R \cap \ker t \leq F$, so that $(u_i m_i, tu_i m_i) \in I \leq K$, from which we get $z_i m_i \in K$.

Now $(x, tx) = z_1 m_1 + z_2 m_2 \in K$. Since $x \in R \cap t^{-1}R$, we also have $(x, tx) \in J$, whence $(x, tx) \in I$. Thus $x \in F$, which is a contradiction.

(c) Inasmuch as Q is a regular, right self-injective ring, eQ must be a nonsingular injective right Q-module. According to [4, Proposition 1.17], eQe is thus a regular, right self-injective ring. In view of (a) and (b), [4, Proposition 1.16] now says that $Z_r(Re) = 0$ and $S^\circ(Re) = eQe$.

Lemma E. *Let e be an idempotent in R such that eR is a two-sided ideal. If $R(1-e) \in \mathcal{S}(R)$, then all nonsingular right (eRe)-modules are projective.*

Proof. Setting $C = eRe$, we see from Lemma D that $Z_r(C) = 0$ and $S^\circ C = eQe$, where $Q = S^\circ R$. Lemma D also shows that $C = Re$, from which we infer that $r \mapsto re$ is a unital ring map of R onto C. Therefore $C \cong R/R(1-e)$. Since $R(1-e) \in \mathcal{S}(R)$, it follows as in [4, Theorem 5.3] that C is a right perfect ring. According to [1, Theorem P], this means that all flat right C-modules are projective; hence it suffices to show that all nonsingular right C-modules are flat. By [4, Proposition 2.1], this is equivalent to showing that $(S^\circ C)_C$ is flat and that $\mathrm{GWD}(C) \leq 1$. We shall prove this by showing that all right C-submodules of $S^\circ C$ are flat.

Thus consider any $E \leq (eQe)_C$. Noting that ER is a nonsingular right R-module and that $H = R(1-e)$ is a two-sided ideal in $\mathcal{S}(R)$, we obtain from [4, Lemma 5.2] that ER/EH is a projective right (R/H)-module. We have an abelian group epimorphism $f\colon ER \to E$ given by $fx = xe$, and it is easily checked that the kernel of f is EH. Inasmuch as $f(xr) = (fx)(re)$ for all $x \in ER$ and $r \in R$, we conclude that E must be a projective right C-module. Therefore E_C is certainly flat.

Lemma F. *If N is any nilpotent two-sided ideal of R, then N_R is finite dimensional.*

Proof. Setting M equal to the $\mathcal{S}$-closure of N_R in R_R, we see from Lemma A that M is a two-sided ideal in $L^*(R)$ whose left annihilator belongs to $\mathcal{S}(R)$. In view of Lemma C, there exists an idempotent $e \in R$ such that $eR = M$. Then Lemmas D and E say that $Z_r(eRe) = 0$ and that all nonsingular right (eRe)-modules are projective. According to [4, Theorem 2.11], eRe must be finite dimensional as a right module over itself. Letting $Q = S^\circ R$, we have $S^\circ(eRe) = eQe$ by Lemma D; hence [4, Theorem 1.26] says that eQe is a semisimple ring.

There must exist orthogonal idempotents $e_1, \ldots, e_n \in eQe$ such that $e_1 + \cdots + e_n = e$ and each $e_i Q e_i$ is a division ring. Noting that Q is a semiprime ring (because it is regular), we see from [6, Proposition 2, p. 63] that the $e_i Q$ are minimal right ideals of Q. Therefore eQ is a finitely generated semisimple right Q-module. Observing that $S^\circ(eR) = eQ$, we infer from [4, Theorem 1.24] that $(eR)_R$ is finite dimensional, whence N_R is finite dimensional.

Lemma G. *Let P denote the prime radical of R. Then any nonzero submodule of P_R contains a nonzero finite-dimensional submodule.*

Proof. Let T denote the union of all nilpotent two-sided ideals of R, and note that T is a two-sided ideal. Any nonzero submodule A of T_R must have nonzero intersection with some nilpotent two-sided ideal N, and $A \cap N$ is a finite dimensional module by Lemma F. Therefore every nonzero submodule of T_R has a nonzero finite dimensional submodule. To prove that P_R satisfies the same property, it suffices to show that T_R is essential in P_R.

Let H be maximal among those two-sided ideals of R containing T for which T_R is essential in H_R. We claim that $P \leq H$.

Suppose not. Inasmuch as P is contained in every semiprime ideal of R [7, Theorem 4.20], H cannot be a semiprime ideal. Thus there exists a two-sided ideal K, properly containing H, such that $K^2 \leq H$. Due to the maximality of H, T_R is not essential in K_R, from which we infer that H_R is not essential in K_R. Therefore there exists a nonzero element $x \in K$ which has no nonzero right multiples in H. Letting J denote the left annihilator of K, we infer from the equation $K^2 \leq H$ that $x \in J \cap K$. But $J \cap K$ is nilpotent and hence contained in T, from which it follows that $x \in H$, which is impossible.

Therefore $P \leq H$; hence T_R is essential in P_R.

Lemma H. *The ring $Q = S^\circ R$ is a splitting ring.*

Proof. We first show that $Q \otimes_R Q$ is a nonsingular right R-module. According to [2, Theorem 1.6], it suffices to show that for any $a \in Q$, the right ideal $I = \{x \in R | ax \in R\}$ has a finitely generated essential submodule. Using [4,Theorem 4.6 and Proposition 4.8], we see that I is AFG, i.e., that $I/\text{soc}(I)$ is finitely generated. Inasmuch as $\text{soc}\,(R_R) = 0$, it follows that I is finitely generated. Therefore $(Q \otimes_R Q)_R$ is nonsingular. According to [4, Lemma 1.25], it follows that the natural map $Q \otimes_R Q \to Q$ is an isomorphism.

We must show that any short exact sequence E: $0 \to C \to B \to A \to 0$ splits, where A, B, and C are right Q-modules such that C is singular and A is nonsingular. According to [4, Proposition 1.10], C_R is singular and A_R is nonsingular, whence E splits *as a sequence of R-modules*. Thus we obtain a split exact sequence E^*: $0 \to C \otimes_R Q \to B \otimes_R Q \to A \otimes_R Q \to 0$ of right Q-modules. Inasmuch as the natural map $Q \otimes_R Q \to Q$ is an isomorphism, we infer that E^* is naturally isomorphic to E, hence E must split.

Lemma I. *If $Q = S^\circ R$, then* $\text{soc}(Q_Q)$ *is finitely generated.*

Proof. According to [5, Theorem 10], Q is an essential product of a ring with essential right socle and ring with zero right socle. Inasmuch as Q is its own maximal right quotient ring, it follows from [5, Proposition 2] that Q is actually a direct product $H \times K$, where H_H has essential socle and K_K has zero socle. If J denotes the socle of Q_Q, then we see that also $J = \mathrm{soc}(H_H)$, and J_H is essential in H_H.

It follows from Lemma H that H is a splitting ring; hence [4, Corollary 5.4] says that H/J is a semiprimary ring. Since Q is a regular ring, we infer that H/J is also a regular ring, whence H/J is actually a semisimple ring.

Now suppose that J_Q is not finitely generated. Then we can write $J = \bigoplus_{n=1}^{\infty} J_n$, where each J_n is an infinite direct sum of simple modules. Observing that H_H is injective (since Q_Q is injective), we infer that for each positive integer s, there must exist an idempotent $e_s \in H$ such that $e_s H$ is an injective hull for J_s and $(1 - e_s)H$ is an injective hull for $\bigoplus_{n \neq s} J_n$. Note that the idempotents e_s are mutually orthogonal. Inasmuch as $e_s H$ contains the infinite direct sum J_s, it follows that $e_s H$ cannot be semisimple. Thus $e_s H \not\leq J$, i.e., $e_s \notin J$. Therefore the images of the elements e_s in H/J form an infinite orthogonal sequence of nonzero idempotents, which contradicts the fact that H/J is a semisimple ring.

Lemma J. *If P denotes the prime radical of R, then P_R is finite dimensional.*

Proof. Set $Q = S^\circ R$ and $J = \mathrm{soc}(Q_Q)$, and let F denote the sum of all the finite-dimensional submodules of P_R. It follows from Lemma G that F is essential in P_R, from which we infer that $S^\circ F = S^\circ P$. If A is any finite-dimensional submodule of P_R, then [4, Theorem 1.24] says that $S^\circ A$ is a finitely generated semisimple right Q-module; hence $A \leq S^\circ A \leq J$. Thus $F \leq J$, and so $S^\circ P = S^\circ F \leq S^\circ J$.

According to Lemma I, J_Q is finitely generated. Inasmuch as Q is a regular ring, it follows that J is a direct summand of Q, from which we infer that $S^\circ J = J$. Therefore $S^\circ P \leq J$, whence $S^\circ P$ is a finitely generated semisimple right Q-module. By [4, Theorem 1.24], P_R must be finite dimensional.

Theorem 2. *Let R be a right nonsingular splitting ring with* $\mathrm{soc}(R_R) = 0$. *Then R is isomorphic to a formal triangular matrix ring* $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$, *where A is a semiprime ring, C is a left and right artinian ring, and ${}_C B$ is faithful.*

Proof. We first need a maximal element in the collection $\mathcal{A}$ of those two-sided ideals in $L^*(R)$ whose left annihilators belong to $\mathcal{S}(R)$. Let P denote the prime radical of R, and set $\mathcal{B} = \{A \cap P \mid A \in \mathcal{A}\}$. Inasmuch as P_R is finite dimensional by Theorem 1, [4, Theorem 1.24] says that $L^*(P)$ has ACC. Thus $\mathcal{B}$ must have a maximal element, which is of the form $M \cap P$ for some $M \in \mathcal{A}$. We claim that M is maximal in $\mathcal{A}$.

Consider any ideal $N \in \mathcal{A}$ which contains M. Letting K denote the left annihilator of N, we see that $N \cap K$ is nilpotent and hence contained in P, whence $N \cap K \leq N \cap P$. The maximality of $M \cap P$ implies that $N \cap P = M \cap P$, from which we obtain $N \cap K \leq M$. Observing that $K \in \mathcal{S}(R)$, we see that $(N \cap K)_R$ is essential in N_R. Inasmuch as $M \in L^*(R)$, it follows that $N \leq M$. Therefore M is maximal in $\mathcal{A}$.

We next claim that R/M is a semiprime ring. If not, then it must have a nonzero nilpotent two-sided ideal N/M. Letting T denote the $\mathcal{S}$-closure of N_R in R_R, it follows just as in Lemma A that the ideal $H = \{r \in R \mid rT \leq M\}$ must belong to $\mathcal{S}(R)$. Inasmuch as the left annihilator K of M also belongs to $\mathcal{S}(R)$, we see that R/K and K/KH are both singular right R-modules, whence R/KH is singular and $KH \in \mathcal{S}(R)$. Noting that $KHT = 0$, we infer that the left annihilator of T belongs to $\mathcal{S}(R)$. But then $T \in \mathcal{A}$, which contradicts the maximality of M.

According to Lemma C, there exists an idempotent $e \in R$ such that $eR = M$. Since eR is a two-sided ideal, we obtain $(1-e)Re = 0$. Therefore R is isomorphic to the ring $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$, where $A = (1-e)R(1-e)$, $B = eR(1-e)$, and $C = eRe$. Observing that $R/M \cong A$, we see that A is a semiprime ring.

Since $M \in \mathcal{A}$, its left annihilator $R(1-e)$ must belong to $\mathcal{S}(R)$. According to Lemmas D and E, $Z_r(C) = 0$ and all nonsingular right C-modules are projective; hence [4, Theorem 2.12] shows that C is left and right artinian. Any $x \in C$ satisfying $xB = 0$ must also satisfy $xR(1-e) = 0$, and then $x = 0$ [because $R(1-e) \in \mathcal{S}(R)$]. Therefore ${}_CB$ is faithful.

3. **Formal triangular matrix rings.** The purpose of this section is to derive a few basic properties of a formal triangular matrix ring $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$. We are mainly interested in when such a ring can be nonsingular, and in finding the maximal quotient ring of such a ring.

Throughout this section, we assume that A and C are rings, that ${}_CB_A$ is a bimodule, and that R is the ring $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$. In order to avoid some unnecessary complications, we also make the stipulation that ${}_CB$ is faithful.

Proposition 3. (a) *A right ideal I of R belongs to $\mathcal{S}(R)$ if and only if it contains a right ideal of the form $\left(\begin{smallmatrix} J & 0 \\ K & 0 \end{smallmatrix}\right)$, where $J \in \mathcal{S}(A)$ and K_A is essential in B_A.*

(b) *R_R is nonsingular if and only if A_A and B_A are both nonsingular.*

(c)
$$\operatorname{soc}(R_R) = \begin{pmatrix} \operatorname{soc}(A_A) & 0 \\ \operatorname{soc}(B_A) & 0 \end{pmatrix}.$$

Proof. (a) If $I \in \mathcal{S}(R)$, then it is easily seen to contain such a right ideal. Conversely, if I contains a right ideal $\left(\begin{smallmatrix} J & 0 \\ K & 0 \end{smallmatrix}\right)$ of the form described, then we easily infer that $\left(\begin{smallmatrix} J & 0 \\ K & 0 \end{smallmatrix}\right)$ is essential in $\left(\begin{smallmatrix} A & 0 \\ B & 0 \end{smallmatrix}\right)$. Inasmuch as ${}_CB$ is faithful, it follows that $\left(\begin{smallmatrix} A & 0 \\ B & 0 \end{smallmatrix}\right) \in \mathcal{S}(R)$, from which we infer that $\left(\begin{smallmatrix} J & 0 \\ K & 0 \end{smallmatrix}\right) \in \mathcal{S}(R)$, and then that $I \in \mathcal{S}(R)$.

(b) If R_R is nonsingular, then it is immediate from (a) that A_A and B_A are nonsingular. Conversely, assume that A_A and B_A are nonsingular, and consider any element $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in Z_r(R)$. In view of (a), we obtain $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)\left(\begin{smallmatrix} J & 0 \\ K & 0 \end{smallmatrix}\right) = 0$ for some $J \in \mathcal{S}(A)$ and some essential submodule K of B. We have $aJ = 0$ and $bJ = 0$; hence $a = 0$ and $b = 0$. We also have $cK = 0$, whence cB is an epimorphic image of the singular module B/K. It follows that $cB = 0$, and then the faithfulness of ${}_CB$ implies that $c = 0$. Therefore $Z_r(R) = 0$.

(c) According to [4, Corollary 1.3], the socle of any module is the intersection of its essential submodules; hence (c) follows immediately from (a).

Let us now assume that A_A and B_A are nonsingular, so that R_R is nonsingular. Set $T = \mathrm{End}_A(S^\circ B)$ and $X = \mathrm{Hom}_A(S^\circ B, S^\circ A)$. Since ${}_C B$ is faithful, we may think of C as a subring of the endomorphism ring of B_A. Then each element $c \in C$ induces a unique endomorphism $S^\circ c$ of $S^\circ B$; hence we obtain an embedding $c \mapsto S^\circ c$ of $C \to T$. For ease of notation, we may thus assume that C is a unital subring of T satisfying $CB \leq B$. We note that the faithfulness of ${}_C B$ is now a consequence of the assumption that C is a subring of T.

Proposition 4. $S^\circ R = \left(\begin{smallmatrix} S^\circ A & X \\ S^\circ B & T \end{smallmatrix}\right)$.

[*Note.* To multiply an element $b \in S^\circ B$ by an element $f \in X$, we just let bf stand for the map $x \mapsto b(fx)$.]

Proof. Set $Q = \left(\begin{smallmatrix} S^\circ A & X \\ S^\circ B & T \end{smallmatrix}\right)$. Recalling that the $S^\circ A$-homomorphisms from $S^\circ A$ to $S^\circ A$ or to $S^\circ B$ are the same as the A-homomorphisms, we see that $S^\circ A$ and $S^\circ B$ may be identified with $\mathrm{End}_A(S^\circ A)$ and $\mathrm{Hom}_A(S^\circ A, S^\circ B)$. With these identifications, Q is naturally isomorphic to the ring $\mathrm{End}_A(S^\circ A \otimes S^\circ B)$. Inasmuch as $S^\circ A \otimes S^\circ B$ is a nonsingular injective right A-module, it follows from [4, Proposition 1.17] that Q is regular and right self-injective.

Inasmuch as A is essential in $S^\circ A$ and B is essential in $S^\circ B$, we see that R_R is essential in the module $P_R = \left(\begin{smallmatrix} S^\circ A & 0 \\ S^\circ B & C \end{smallmatrix}\right)$. Now P is also a unital subring of Q, and we check that P_P is essential in Q_P, from which it is easy to infer that R_R is essential in Q_R. According to [4, Proposition 1.16], it follows that $S^\circ R = Q$.

4. Triangular splitting rings. This section is devoted to developing necessary and sufficient conditions for a formal triangular matrix ring (with zero socle) to be a splitting ring. In light of §3, we assume throughout this section that:

(a) A is a right nonsingular ring.
(b) B is a nonsingular right A-module.
(c) C is a unital subring of $T = \mathrm{End}_A(S^\circ B)$ such that $CB \leq B$.
(d) $R = \left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$.

For convenience, we label the following three two-sided ideals of R:

$$R_{12} = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad \text{and} \quad R_{23} = \begin{pmatrix} 0 & 0 \\ B & C \end{pmatrix}.$$

Note that ${}_R(R/R_{12})$ and $(R/R_{23})_R$ are projective, that $R_{12} \in \mathcal{S}(R)$, and that $R_{23} \in L^*(R)$.

Theorem 5. *Assume that* $\mathrm{soc}(R_R) = 0$. *If R is a splitting ring, then*

(a) *A is a splitting ring.*
(b) *B_A is injective.*
(c) *C_C is essential in T_C.*
(d) *All nonsingular right C-modules are projective.*

(For characterizations of rings satisfying (d), see [4, Theorems 2.11, 2.12, and 2.15].)

Proof. (a) Note that $A \cong R/R_{23}$. Since R_{23} is a two-sided ideal in $L^*(R)$, [4, Proposition 1.11] says that the singular submodule of any right (R/R_{23})-module is the same whether considered as an (R/R_{23})-module or as an R-module. Thus R/R_{23} must be a splitting ring.

(c)(d) The element $e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is an idempotent in R such that eR is a two-sided ideal and such that $R(1-e) \in \mathcal{S}(R)$. In view of Proposition 4, we see that $eRe = \left(\begin{smallmatrix} 0 & 0 \\ 0 & C \end{smallmatrix}\right)$ and $e(S^\circ R)e = \left(\begin{smallmatrix} 0 & 0 \\ 0 & T \end{smallmatrix}\right)$. According to Lemmas D and E, it follows that C_C is essential in T_C and that all nonsingular right C-modules are projective. For use in the proof of (b), we note that Lemma D also says that $Z_r(C) = 0$ and that $S^\circ C = T$.

(b) We proceed via several lemmas. With the exception of Lemma N, we stipulate that Lemmas K through O all include the hypothesis that R is a splitting ring with $\mathrm{soc}(R_R) = 0$.

Lemma K. *$S^\circ B$ is a finitely generated semisimple right $S^\circ A$-module.*

Proof. Since R_2 is nilpotent and therefore contained in the prime radical of R, Theorem 1 says that R_2 is a finite-dimensional right R-module. Thus B_A must be finite dimensional. According to [4, Theorem 1.24], $S^\circ B$ is finitely generated and semisimple.

Lemma L. *If M is any simple $S^\circ A$-submodule of $S^\circ B$, then there exists an idempotent $e \in C$ such that $e(S^\circ B) = M$ and $eTe = eCe$.*

Proof. In view of Lemma K, M must be a direct summand of $S^\circ B$; hence $M = f(S^\circ B)$ for some idempotent $f \in T$. Inasmuch as the A-endomorphisms of M coincide with the $S^\circ A$-endomorphisms, we infer that fTf is isomorphic to the ring of $S^\circ A$-endomorphisms of M, from which it follows that fTf is a division ring. Noting that T is regular and therefore semiprime, we obtain from [6, Proposition 2, p. 63] that fT is a minimal right ideal of T.

Observing that $C/(C \cap fT)$ is a nonsingular right C-module, we see from (d) that $C \cap fT = eC$ for some idempotent $e \in C$. Since C_C is essential in T_C by (c), we obtain $C \cap fT \neq 0$, whence the minimality of fT implies that $eT = fT$. Therefore $e(S^\circ B) = f(S^\circ B) = M$. Since eT is a minimal right ideal of T, it follows as in the proof of [4, Theorem 2.14] that eT must be a uniserial right C-module. In particular, eC must be a characteristic submodule of eT, from which we infer that $(eTe)(eC) \leq eC$; hence $eTe = eCe$.

Lemma M. *Let M be any simple $S^\circ A$-submodule of $S^\circ B$. If $I = \{a \in A \mid (M \cap B)a = 0\}$, then $MI = 0$.*

Proof. Set $Q = S^\circ R$ and $H = \left(\begin{smallmatrix} I & 0 \\ B & 0 \end{smallmatrix}\right)$, and note that H is a two-sided ideal of R. Since $S^\circ I$ is injective, we must have $S^\circ I = f(S^\circ A)$ for some idempotent $f \in S^\circ A$. Setting $g = \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we infer that H_R is essential in gQ, from which it follows that $S^\circ H = gQ$. Thus $R \cap gQ$ is the $\mathcal{S}$-closure of H_R in R_R, which is a two-sided ideal of R.

According to Lemma D, Rg is a unital subring of gQg and Rg is an essential right (Rg)-submodule of gQg. Observing that

$$Rg = \begin{pmatrix} Af & 0 \\ Bf & C \end{pmatrix} \quad \text{and} \quad gQg = \begin{pmatrix} f(S^\circ A)f & fX \\ (S^\circ B)f & T \end{pmatrix},$$

we infer that Af is a unital subring of $f(S^\circ A)f$ and that Bf is an essential right (Af)-submodule of $(S^\circ B)f$.

According to Lemma L, there exists an idempotent $e \in C$ such that $e(S^\circ B) = M$. Since $e \in C$, we have $eB \leq B$, whence $eB = M \cap B$. Thus $eBI = 0$; hence $eB(S^\circ I)$ is a sum of epimorphic images of the singular module $S^\circ I/I$. It follows that $eB(S^\circ I) = 0$, whence $eBf = 0$. Observing that eBf is an essential right (Af)-submodule of $e(S^\circ B)f$, we obtain $e(S^\circ B)f = 0$, from which we conclude that $MI = 0$.

Lemma N. [*For this lemma, we only need the hypotheses that* $\operatorname{soc}(R_R) = 0$ *and that* A *is a splitting ring.*] *Let* M *be any simple* $S^\circ A$*-submodule of* $S^\circ B$. *If* $I = \{a \in A \mid Ma = 0\}$, *then* $Z_r(A/I) = 0$, M *is a simple right* $S^\circ(A/I)$*-module, and* $S^\circ(A/I)$ *is a simple artinian ring.*

Proof. Inasmuch as M is nonsingular, we see that $I \in L^*(A)$. According to [4, Proposition 1.11], it follows that $Z_r(A/I) = 0$ and that the singular submodule of any right (A/I)-module is the same whether considered as an (A/I)-module or as an A-module. Since A is a splitting ring, it follows that A/I is a splitting ring also. By Proposition 3, we have $\operatorname{soc}(A_A) = 0$. Recalling that all simple nonsingular modules are projective [4, pp. 55, 56], we infer that the ring A/I must have zero right socle. Setting $P = S^\circ(A/I)$, we thus obtain from Lemma I that $\operatorname{soc}(P_P)$ is finitely generated. Inasmuch as P is a regular ring, $\operatorname{soc}(P_P)$ must thus be a direct summand of P_P.

Noting that M is finitely generated as an $S^\circ A$-module, we see from [4, Proposition 1.15] that M is a direct summand of $S^\circ B$, from which it follows that M_A is injective. Since $MI = 0$, M is also an injective right (A/I)-module. Now M is nonsingular as an A-module and thus as an (A/I)-module; hence we obtain $S^\circ_{A/I} M = M$. Therefore M is a right P-module.

Inasmuch as $S^\circ_A M = M$, the simplicity of M implies that M is indecomposable as an A-module, from which we infer that M must also be indecomposable as a P-module. Noting from [4, Proposition 1.15] that all finitely generated P-submodules of M are direct summands of M, we conclude that M is a simple P-module.

Observing that $\{x \in P \mid Mx = 0\} \cap (A/I) = 0$, we obtain $\{x \in P \mid Mx = 0\} = 0$. Therefore M is a faithful simple P-module; hence P is a primitive ring. In particular, P is a prime ring. The module M_P is also nonsingular and simple, hence projective, from which we see that P must contain a minimal right ideal. Thus $\mathrm{soc}(P_P)$ is a nonzero two-sided ideal in the prime ring P; hence the left annihilator of $\mathrm{soc}(P_P)$ must be zero. It follows that $\mathrm{soc}(P_P)$ is an essential right ideal of P. Since $\mathrm{soc}(P_P)$ is also a direct summand of P, we conclude that $\mathrm{soc}(P_P) = P$, i.e., P is a semisimple ring. Inasmuch as P is prime, it must therefore be simple artinian.

Lemma O. *B_A is injective.*

Proof. Suppose not. Since $S°B$ is an injective A-module, we obtain $B < S°B$. In view of Lemma K, $S°B$ must contain a simple $S°A$-submodule M such that $M \nleq B$, i.e., $M \cap B < M$. Setting $I = \{a \in A \mid (M \cap B)a = 0\}$ and $P = S°(A/I)$, we see from Lemmas M and N that $MI = 0$, $Z_r(A/I) = 0$, M is a simple right P-module, and P is a simple artinian ring. Note that since $\{x \in P \mid (M \cap B)x = 0\} \cap (A/I) = 0$, we obtain $\{x \in P \mid (M \cap B)x = 0\} = 0$.

According to Lemma L, there exists an idempotent $e \in C$ such that $e(S°B) = M$ and $eTe = eCe$. Inasmuch as $CB \leq B$, we infer that $M \cap B$ is a left (eTe)-submodule of M.

The P-endomorphisms of M coincide with the (A/I)-endomorphisms, and the A-endomorphisms of M are just the left multiplications by the elements of eTe; hence we may identify eTe with the endomorphism ring of M_P. Since M_P is simple and P is a simple artinian ring, we infer that eTe is a division ring, that M is a finite-dimensional left vector space over eTe, and that P is the ring of all linear transformations on M. However, $M \cap B$ is a proper subspace of M, and no nonzero element of P annihilates $M \cap B$, which is impossible.

Theorem 6. *Assume that* $\mathrm{soc}(R_R) = 0$. *If the following conditions are satisfied, then R is a splitting ring*:

(a) *A is a splitting ring.*

(b) *B_A is injective.*

(c) *C_C is essential in T_C.*

(d) *All nonsingular right C-modules are projective.*

Proof. Once again we organize the proof as a series of lemmas. We stipulate that each of Lemmas P through U contains conditions (a)–(d) in its hypotheses.

Lemma P. *Any direct sum of copies of B_A is injective.*

Proof. Since T is the endomorphism ring of the nonsingular injective module $S°B$, [4, Proposition 1.17] says that T is a regular, right self-injective ring. In light of condition (c), we see from [4, Proposition 1.16] that $Z_r(C) = 0$ and $S°C = T$. According to [4, Theorem 2.11], condition (d) implies that C_C is finite dimensional, whence [4, Theorem 1.26] says that T is a semisimple ring.

Now there exist orthogonal idempotents $e_1, \ldots, e_n \in T$ such that $e_1 + \cdots + e_n = 1$ and each $e_i T e_i$ is a division ring. Observing from condition (b) that $S^\circ B = B$, we see that $B = e_1 B \otimes \cdots \otimes e_n B$, and that each $e_i B$ is an indecomposable A-module, hence an indecomposable $S^\circ A$-module. According to [4, Proposition 1.15], every finitely generated $S^\circ A$-submodule of B is a direct summand of B, from which we infer that the modules $e_i B$ are simple $S^\circ A$-modules.

To show that any direct sum of copies of B_A is injective, it suffices to show that any direct sum of copies of M_A is injective, where M is any one of the modules $e_i B$. Setting $I = \{a \in A \mid Ma = 0\}$, we obtain from Lemma N that $Z_r(A/I) = 0$ and that $S^\circ(A/I)$ is a simple artinian ring. According to [4, Theorem 1.26], all direct sums of nonsingular injective (A/I)-modules are injective. Inasmuch as M is a nonsingular injective A-module, it must also be a nonsingular injective (A/I)-module; hence any direct sum $\bigoplus M_i$ of copies of M must be injective as an (A/I)-module. Noting that $\prod M_i$ is an (A/I)-module, we infer that $\bigoplus M_i$ is a direct summand of $\prod M_i$, from which it follows that $\bigoplus M_i$ is injective as an A-module.

Lemma Q. *If N is any nonsingular right R-module, then NR_{23} is a direct summand of N.*

Proof. The module NR_2 must be isomorphic to F/K for some direct sum F of copies of R_2 and some $K \in L^*(F)$. In view of Lemma P, we infer that F is injective as a right (R/R_{23})-module. Inasmuch as $K \in L^*(F)$, it follows that K must be a direct summand of F, and thus that NR_2 is injective as an (R/R_{23})-module. Noting that NR_{12} is an (R/R_{23})-module which contains NR_2, we conclude that $NR_{12} = NR_2 \otimes W$ for some W.

Since R_2 is essential in R_{23}, it follows that R_{23}/R_2 is a singular right R-module. Noting that NR_{23}/NR_2 is a sum of epimorphic images of R_{23}/R_2, we see that NR_{23}/NR_2 is singular. Inasmuch as NR_{23} is nonsingular, it follows that NR_2 is essential in NR_{23}. We now take the equation $NR_2 \cap W = 0$ and infer from this that $NR_{23} \cap W = 0$. Checking that $N = NR_{23} + NR_{12} = NR_{23} + W$, we conclude that $N = NR_{23} \otimes W$.

Lemma R. *If N is any nonsingular right R-module, then N/NR_{12} is a projective right (R/R_{12})-module.*

Proof. In view of condition (d), it suffices to show that N/NR_{12} is nonsingular as an (R/R_{12})-module. Since N_R is nonsingular, there exists a monomorphism $N \to \prod Q_i$, where each Q_i is a copy of $Q = S^\circ R$. Inasmuch as ${}_R(R/R_{12})$ is finitely generated and projective, we obtain another monomorphism

$$N \otimes_R (R/R_{12}) \to (\prod Q_i) \otimes_R (R/R_{12}) \to \prod [Q \otimes_R (R/R_{12})].$$

Thus N/NR_{12} is embedded in a direct product of copies of Q/QR_{12}; hence it suffices to show that Q/QR_{12} is nonsingular as an (R/R_{12})-module. Using Proposition 4 to check that

$$Q/QR_{12} = \begin{pmatrix} S^\circ A & X \\ S^\circ B & T \end{pmatrix} \Big/ \begin{pmatrix} S^\circ A & 0 \\ S^\circ B & 0 \end{pmatrix},$$

we infer that it suffices to prove that X_C and T_C are nonsingular.

As in Lemma P, we have $Z_r(C) = 0$ and $S^\circ C = T$, whence T_C is nonsingular. Now consider any element $f \in Z(X_C)$. Since f maps $S^\circ B$ into the nonsingular module $S^\circ A$, we have $\ker f \in L^*(S^\circ B)$. Inasmuch as $S^\circ B$ is injective, it follows that $\ker f = e(S^\circ B)$ for some indempotent $e \in T$. We now infer that $fT \cong (1-e)T$, from which it follows that $(fT)_C$ is nonsingular, and thus $f = 0$. Therefore $Z(X_C) = 0$.

Lemma S. *Let n be any positive integer, and let $K \in L^*(B^n)$. If $J = \{x \in C^n \mid xB \leq K\}$, then $JB = K$.*

Proof. As in Lemma P, we have $Z_r(C) = 0$ and $S^\circ C = T$. In light of condition (d), we see from [4, Theorem 2.5] that C is right semihereditary and that $Z[(T \otimes_C T)_C] = 0$. Then [4, Lemma 2.2] says that ${}_C T$ is flat, while [4, Lemma 1.25] shows that the natural map $T \otimes_C T \to T$ is an isomorphism.

Inasmuch as B_A is injective, we have $B = S^\circ B$; hence B is a left T-module. Then ${}_T B$ is flat because T is a regular ring, and we infer from the flatness of ${}_C T$ that ${}_C B$ must be flat.

Setting $L = \{x \in T^n \mid xB \leq K\}$, we note that L is a right T-submodule of T^n. We have a monomorphism $C^n/J \to T^n/L$, from which we obtain another monomorphism $(C^n/J) \otimes_C B \to (T^n/L) \otimes_C B$. Now $(C^n/J) \otimes_C B$ is naturally isomorphic to B^n/JB, and we also have natural isomorphisms

$$(T^n/L) \otimes_C B \to (T^n/L) \otimes_T T \otimes_C T \otimes_T B \to (T^n/L) \otimes_T T \otimes_T B$$
$$\to (T^n/L) \otimes_T B \to B^n/LB;$$

hence we conclude that the natural map $B^n/JB \to B^n/LB$ is injective. Therefore $JB = LB$.

Inasmuch as B^n is injective and $K \in L^*(B^n)$, K must be a direct summand of B^n. Thus there exists an idempotent $n \times n$ matrix p over T such that $pB^n = K$. Given any $x \in K$, we can obtain $x = u_1 b_1 + \cdots + u_n b_n$ for appropriate choices of $u_i \in T^n$ and $b_i \in B$. Since each $pu_i B \leq pB^n = K$, we see that each $pu_i \in L$. Observing that $x = px$, it follows that $x \in LB$. Therefore $K = LB = JB$.

Lemma T. *If N is any nonsingular right R-module, then* $\operatorname{Tor}_1^R(N, R/R_{23}) = 0$.

Proof. We may assume, without loss of generality, that N is finitely generated, and we shall prove that the map $N \otimes_R R_{23} \to N$ is injective. Inasmuch as NR_{23} is a direct summand of N by Lemma Q, the map f: $NR_{23} \otimes_R R_{23} \to N \otimes_R R_{23}$ is injective. Noting that R_{23} is idempotent, we see that f is also surjective and hence an isomorphism. Thus it suffices to prove that $NR_{23} \otimes_R R_{23} \to NR_{23}$ is injective.

Since N is finitely generated, we obtain $NR_{23} \cong R_{23}^n/H$ for some positive integer n and some $H \in L^*(R_{23}^n)$. We check that $H = \left(\begin{smallmatrix} 0 & 0 \\ K & J \end{smallmatrix}\right)$ for some $K \in L^*(B^n)$ and some $J \leq C^n$. Since H is a submodule of R_{23}^n, we must have $JB \leq K$. Given any $x \in C^n$ for which $xB \leq K$, we see that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right) R_{12} \leq H$. Inasmuch as $R_{12} \in \mathscr{S}(R)$, it follows that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right) \in H$, whence $x \in J$. Therefore $J = \{x \in C^n \mid xB \leq K\}$; hence according to Lemma S we obtain $JB = K$.

Now $HR_{23} = \left(\begin{smallmatrix} 0 & 0 \\ JB & J \end{smallmatrix}\right) = H$; hence the map $H/HR_{23} \to R_{23}^n/(R_{23}^n)R_{23}$ is injective. Inasmuch as R_{23}^n is projective, it follows that $\mathrm{Tor}_1^R(R_{23}^n/H, R/R_{23}) = 0$, i.e., $\mathrm{Tor}_1^R(NR_{23}, R/R_{23}) = 0$. Therefore $NR_{23} \otimes_R R_{23} \to NR_{23}$ is injective.

Lemma U. *R is a splitting ring.*

Proof. We must show that $\mathrm{Ext}_R^1(N, W) = 0$ whenever N is a nonsingular right R-module and W is a singular right R-module. Since it suffices to show that $\mathrm{Ext}_R^1(N, W/WR_{12}) = 0$ and $\mathrm{Ext}_R^1(N, WR_{12}) = 0$, we may assume that either $WR_{12} = 0$ or $WR_{23} = 0$.

Case I. $WR_{12} = 0$. Consider any short exact sequence $E: 0 \to W \to V \to N \to 0$ of right R-modules. Since ${}_R(R/R_{12})$ is projective, we obtain another exact sequence $E^*: 0 \to W \to V/VR_{12} \to N/NR_{12} \to 0$. According to Lemma R, N/NR_{12} is a projective right (R/R_{12})-module; hence E^* splits, from which we infer that E splits.

Case II. $WR_{23} = 0$. Consider any short exact sequence $E: 0 \to W \to V \to N \to 0$ of right R-modules. Noting from Lemma T that $\mathrm{Tor}_1^R(N, R/R_{23}) = 0$, we obtain another exact sequence $E^*: 0 \to W \to V/VR_{23} \to N/NR_{23} \to 0$. Inasmuch as R_{23} is a two-sided ideal in $L^*(R)$, [4, Proposition 1.11] says that the singular submodule of any right (R/R_{23})-module is the same whether considered as an (R/R_{23})-module or as an R-module. In particular, W must be a singular (R/R_{23})-module. Considering that NR_{23} is a direct summand of N by Lemma Q, we see that N/NR_{23} is nonsingular as an R-module and hence as an (R/R_{23})-module. Inasmuch as R/R_{23} is a splitting ring by (a), it follows that E^* splits, from which we conclude that E splits.

5. **Conclusion.** Combining Theorems 2, 5, and 6, we obtain the following structure theorem for splitting rings with zero socle:

Theorem 7. *Let R be a right nonsingular ring with zero socle. Then R is a splitting ring if and only if R is isomorphic to a ring of the form* $\left(\begin{smallmatrix} A & 0 \\ B & C \end{smallmatrix}\right)$, *where*

(a) *A is a semiprime right nonsingular splitting ring.*

(b) *B is a nonsingular injective right A-module.*

(c) *C is a unital subring of $T = \mathrm{End}_A(B)$.*

(d) *C_C is essential in T_C.*

(e) *All nonsingular right C-modules are projective.*

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